Imaging Shear Stiffness Tissue Properties Using Inverse Methods When Measurements Are Time Dependent.

Joyce R. McLaughlin

Rensselaer Polytechnic Institute

Tutorial

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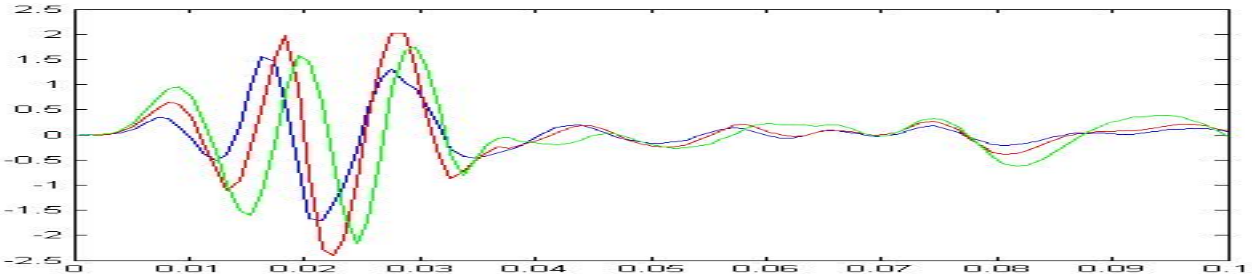
Given:

1) Displacement data on a set of grid points on an image plane or a sequence of closely spaced image planes.

2) One or more components of the elastic vector are measured.

3) Either : time traces of the displacement data are given;
or : amplitude of a time harmonic excitation is given.



Use :

(1) Mathematical model that governs the displacement;

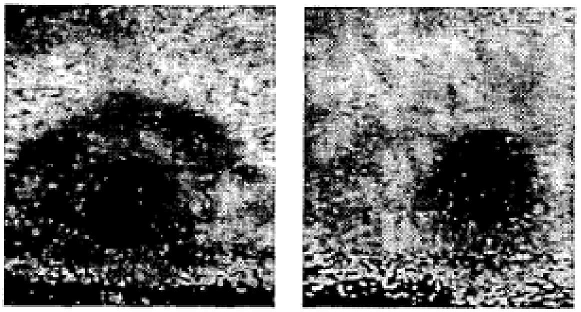
(2) Smart algorithms that maximize information retrieval from the data.

To :

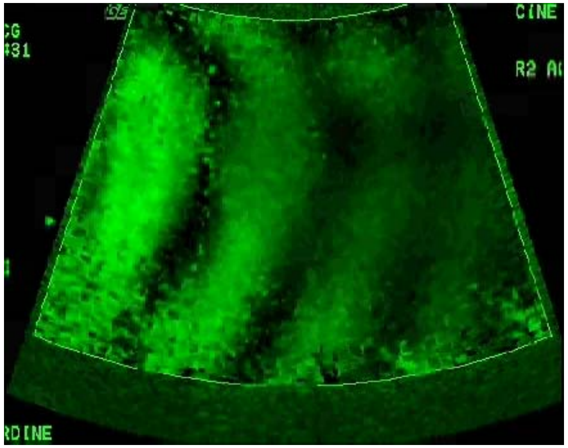
(1) Create useful images that identify small inclusions or low contrast changes or new information rich shear wave related physical properties;

(2) Motivate new applications or, in some cases, design of experiment.

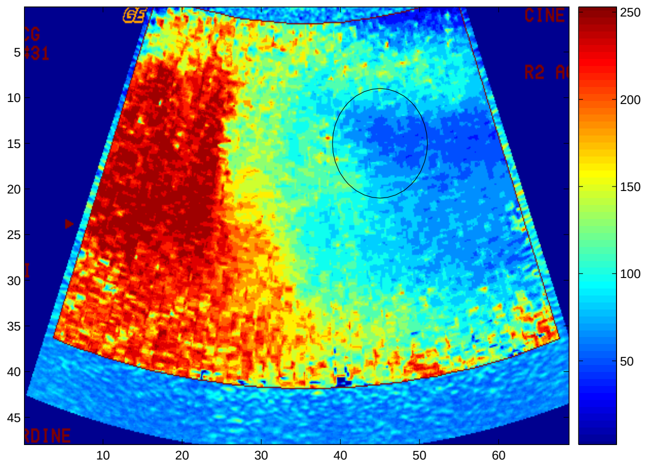
Images of University of Rochester Data (Parker)



Sonoelastography



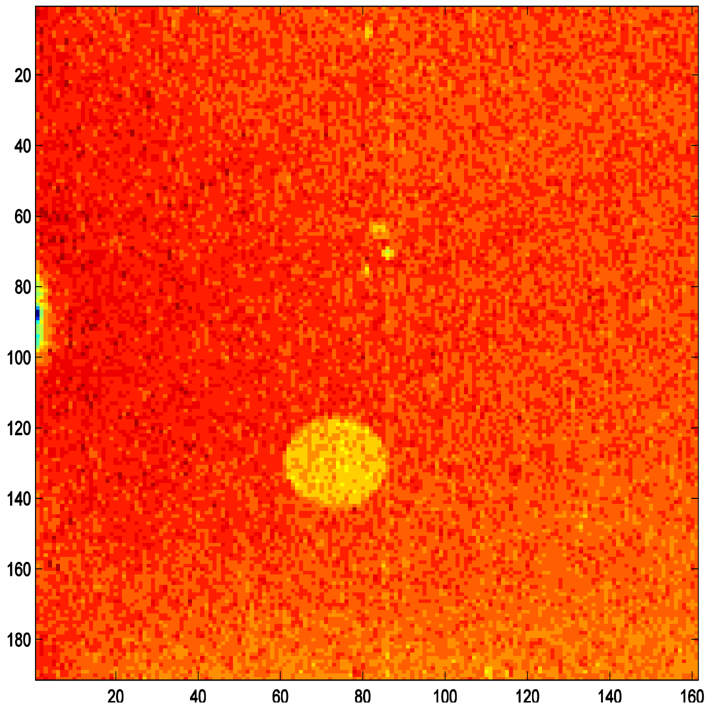
Holographic Wave Movie



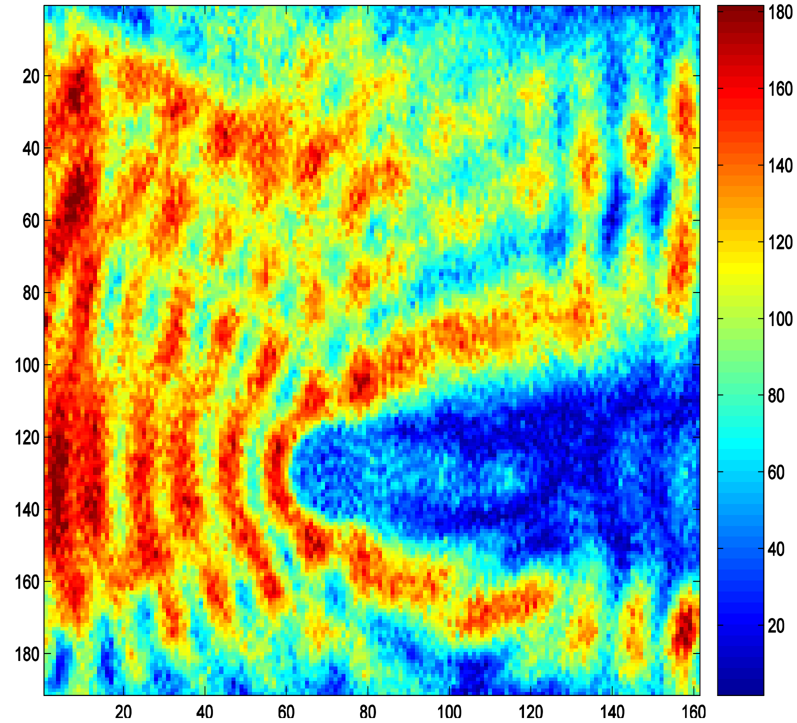
Amplitude Data

Bscan

MR Data (R. Ehman, Mayo Clinic)

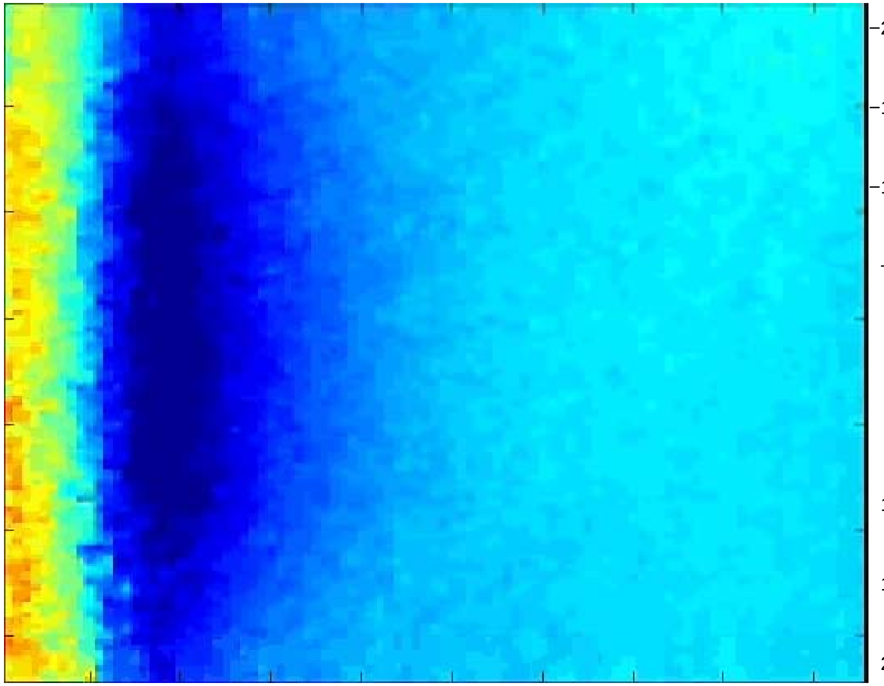


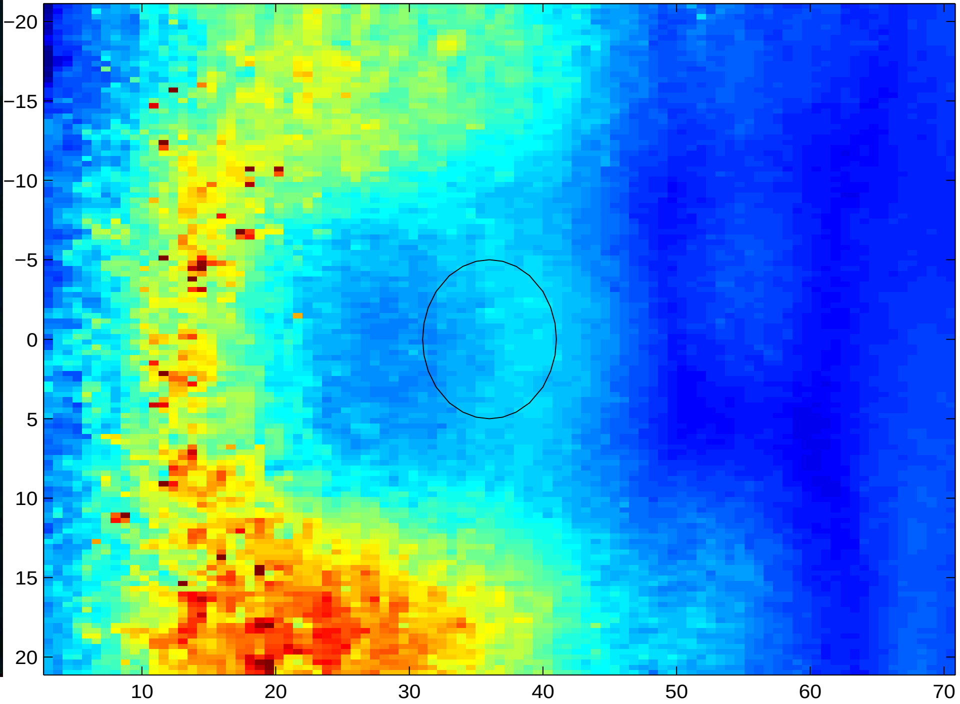
MR Image of Phantom



Elastic Wave Amplitude Image

Fink (ESPCI) Data





Movie of Displacement Data from Transient Elastography Experiment

Amplitude Image

Give models for $\underline{u}$

Isotropic Elastic Model – time dependent Given $\underline{u}(x,t)$

$$\nabla(\lambda\nabla\cdot\underline{u}) + \nabla\cdot(\mu(\nabla\underline{u} + (\nabla\underline{u})^T)) = \rho\underline{u}_{tt}$$

single frequency Given $\hat{\underline{u}}(x,\omega)$

$$\nabla(\lambda\nabla\cdot\hat{\underline{u}}) + \nabla\cdot(\mu(\nabla\hat{\underline{u}} + (\nabla\hat{\underline{u}})^T)) + \omega^2\rho\hat{\underline{u}} = 0$$

wave equation – time dependent Given $u(x,t)$

$$\nabla\cdot(\mu\nabla u) = \rho u_{tt}$$

single frequency Given $\hat{u}(x,\omega)$

$$\nabla\cdot(\mu\nabla\hat{u}) + \omega^2\rho\hat{u} = 0$$

Notation: $\nabla\cdot\underline{u} = \frac{\partial}{\partial x_1}u_1 + \frac{\partial}{\partial x_2}u_2 + \frac{\partial}{\partial x_3}u_3,\quad \nabla f = (f_{x_1}, f_{x_2}, f_{x_3})$

A. Locally constant assumption:

Isotropic elastic model
- time dependent $\nabla p + \mu[\triangle \underline{u} + \nabla(\nabla\cdot\underline{u})] = \rho \underline{u}_{tt}, \quad p = \lambda \nabla\cdot\underline{u},$
- single frequency $\nabla \hat{p} + \mu[\triangle \hat{\underline{u}} + \nabla(\nabla\cdot\hat{\underline{u}})] + \omega^2\rho\hat{\underline{u}} = 0$
- time dependent wave equation $\mu\triangle u = \rho u_{tt}$
- single frequency $\mu\triangle\hat{u} + \omega^2\rho\hat{u} = 0$

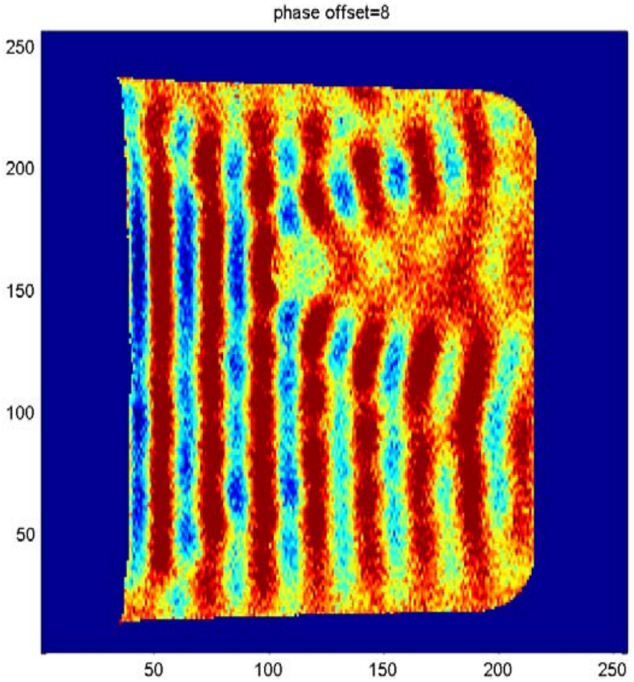
B. Geometric Optics Approximation.

$$u = [M_0 + \frac{1}{i\omega}M_1 + \cdots]e^{i\omega\phi}, \omega \gg 1$$

$$\sqrt{\mu/\rho}|\nabla\phi| = 1$$

$$\sqrt{\mu/\rho} = \text{phase wave speed}$$

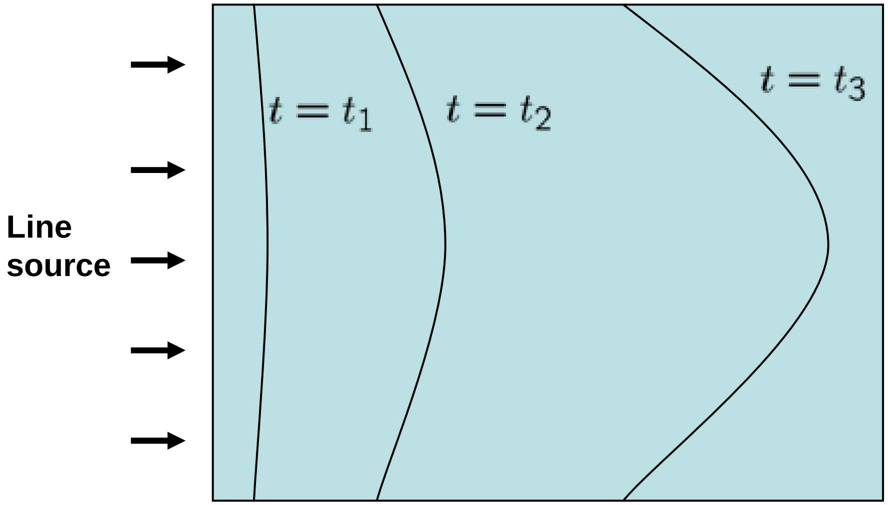
C. Incompressible-Volume Preserving - $\nabla\cdot\underline{u} = 0$



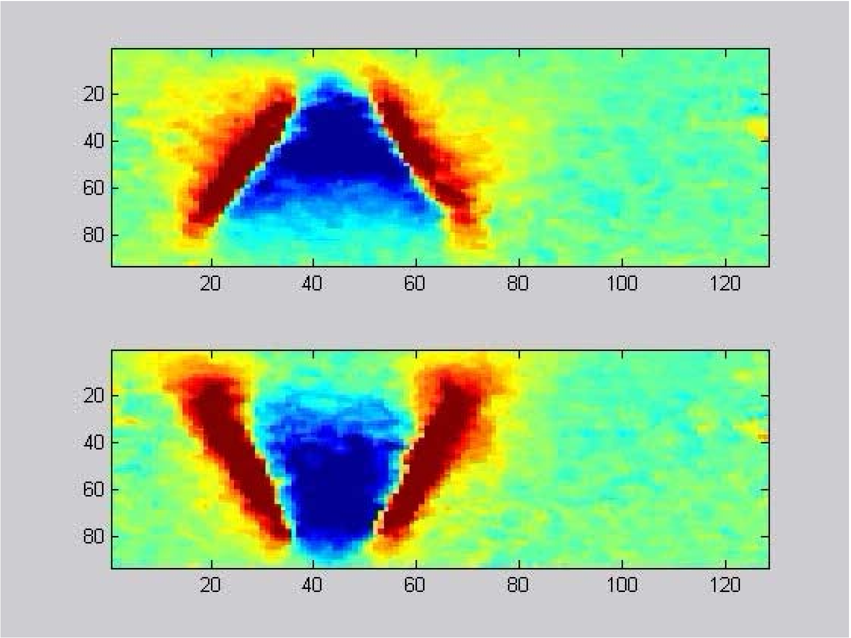
Notation: $\triangle u = \frac{\partial^2}{\partial x_1^2}u + \frac{\partial^2}{\partial x_2^2}u + \frac{\partial^2}{\partial x_3^2}u$

- Wave equation: $\rho u_{tt} = \nabla \cdot (\mu \nabla u)$

- Propagating front:



Transient Elastography



Supersonic Imaging

- Let $\widehat{T}(x) = \min\{t : u(x,s) = 0, s \leq t\}$ - Arrival Time.

$$\sqrt{\mu/\rho}\,|\nabla \widehat{T}| = 1$$

Consider:

$$\nabla \cdot (\mu \nabla u) = \rho u_{tt}$$

$$\nabla \mu \cdot \nabla \hat{u} + \mu \triangle \hat{u} + \omega^2 \rho \hat{u} = \nabla \cdot (\mu \nabla \hat{u}) + \omega^2 \rho \hat{u} = 0$$

locally constant assumption yields

$$\frac{\mu}{\rho} = \frac{-\omega^2 \hat{u}}{\triangle \hat{u}}$$

or target phase wave speed or arrival time equation (Eikonal equation)

$$\sqrt{\mu/\rho}|\nabla \phi| = 1 \qquad \sqrt{\mu/\rho}|\nabla \hat{T}| = 1$$

Finite Difference Methods For $\nabla\cdot(\mu\nabla\hat{u})+\omega^2\rho\hat{u}=0$

1. Calculate the derivatives of the data:

   A. (i) Use centered difference $\hat{u}_x^h(x)=\frac{\hat{u}(x+h)-\hat{u}(x-h)}{2h}$ (too noisy) , stepsize=h;

   (ii) Calculate centered difference for several stepsize $\{h_i\}_{i=1}^n$, Take weighted average or median (see K. Lin's talk) of $\{\hat{u}_x^{h_i}\}_{i=1}^n$. Control noise by bounding the variance.

   (iii) Use similar idea for second derivatives.

   B. (i) Start with A (i) above;

   (ii) Smooth by mollifying $\tilde{u}_x(x)=\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty}e^{-\frac{(y-x)^2}{2\sigma^2}}\hat{u}_x^h(y)dy$ (equivalent to low pass filter).

   (iii) Use similar idea for second derivatives.

1. (cont) Calculate derivatives of the data:

C. (i) Assume $\hat{u} = \sum_{i=1}^{n} a_i \phi_i(x), \{\phi_i\}_{i=1}^{n}$ polynomials or finite elements.

Choose $\{a_i\}_{i=1}^{n}$ to minimize

$$\int_{x_0}^{x_1} \left[\sum_{i=1}^{n} a_i \phi_i(x) - \hat{u}(x)\right]^2 dx$$ (least squares approximation)

(ii) $\hat{u}_x = \sum_{i=1}^{n} a_i \phi_{i,x}(x)$

(iii) Use similar idea for second derivatives.

2. Modify step sizes according to the signal to noise (SNR) ratio and the oscillation of $\hat{u}$.

Calculate μ/ρ or μ

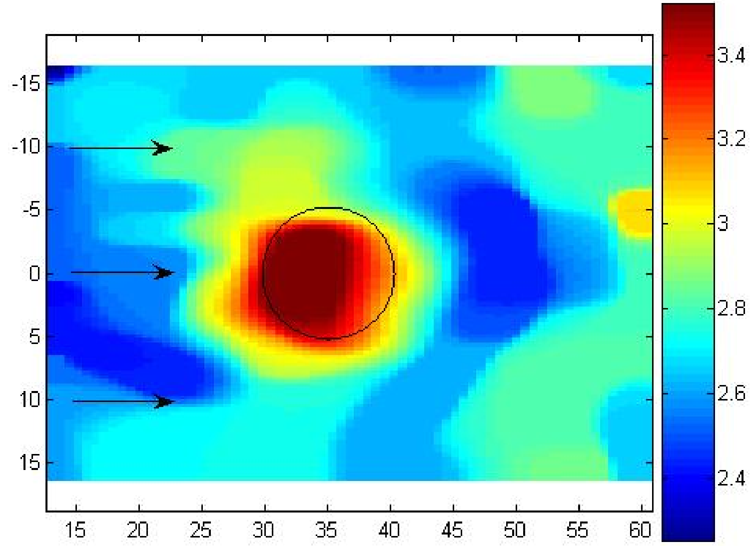
I. Locally constant assumption:

$$\mu/\rho = -\omega^2 u^s / \triangle u^s, \triangle u^s, u^s \text{ smoothed}$$

Advantage : Local algorithm.

Disadvantage : (a) neglect $\nabla\mu$ terms and information of other components of elastic vectors

(b) oscillations in u may restrict noise removal.



Wave Speed Reconstruction With Fink's Isotropic Phantom Data By Method A

II. Currently being investigated:

$$\nabla\mu \cdot \nabla u^s + \mu \triangle u^s + \omega^2 \rho u^s = 0$$

first assume $\rho \equiv$ constant

Basic Questions: (a) Does elimination of $\nabla\mu \cdot \nabla u^s$ term result in serious imaging error?

(b) Can we make a local algorithm in this case?

Calculating Phase Wave Speed or Shear Wave Speed in

$$\sqrt{\mu/\rho} = 1/|\nabla\phi|$$

(Geometric Optics Assumption)

$$\sqrt{\mu/\rho} = 1/|\nabla\hat{T}|$$

(No Locally Constant Assumption)

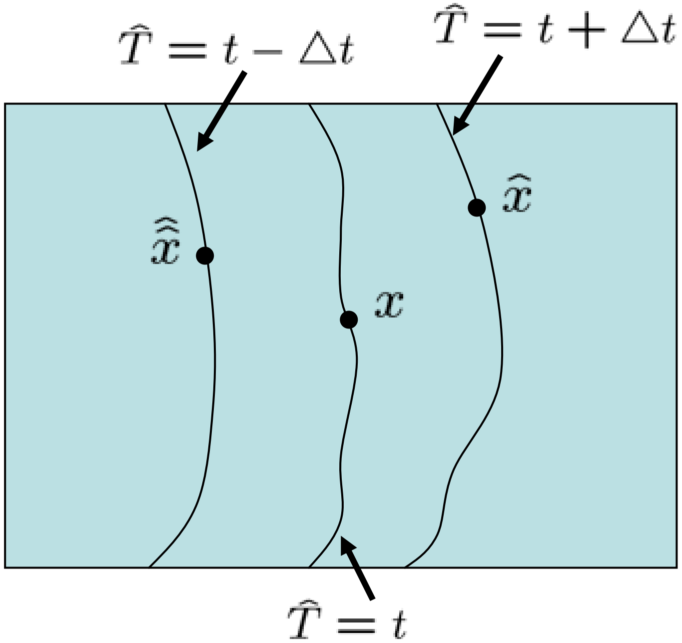
Methods:

1. Make polynomial approximation of ϕ to calculate derivatives;

2. Use averaging or medians of finite difference approximations to get $\nabla\phi$ or $\nabla\hat{T}$;

3. Distance method (McLaughlin, Renzi)
   – uses speed = distance / time

$$\sqrt{\frac{\mu}{\rho}(x)} = \frac{1}{2\triangle t}\Big[\min_{T(\hat{x})=T(x)+\triangle t}|\hat{x}-x| + \min_{T(\hat{\hat{x}})=T(x)-\triangle t}|\hat{\hat{x}}-x|\Big]$$



slow but reliable.
Choose step size according to SNR

Calculating Phase Wave Speed or Shear Wave Speed in

$$\sqrt{\mu/\rho} = 1/|\nabla \phi|$$

(Geometric Optics Assumption)

$$\sqrt{\mu/\rho} = 1/|\nabla \hat{T}|$$

(No Locally Constant Assumption)

4. Level Curve Method (McLaughlin, Renzi)– Fast method *O(N), N* number of grid points.

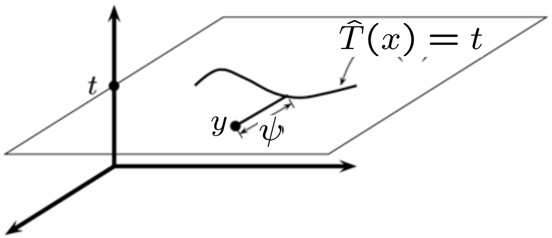
(a) Construct higher dimensional function $\psi(x,t)$;

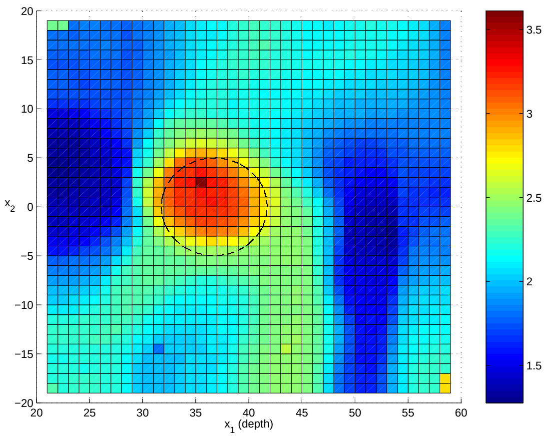
$$\psi(x, \hat{T}(x)) = 0$$

(b) $\psi(y,t) = \pm \min_{T(x)=t} |x-y|$;

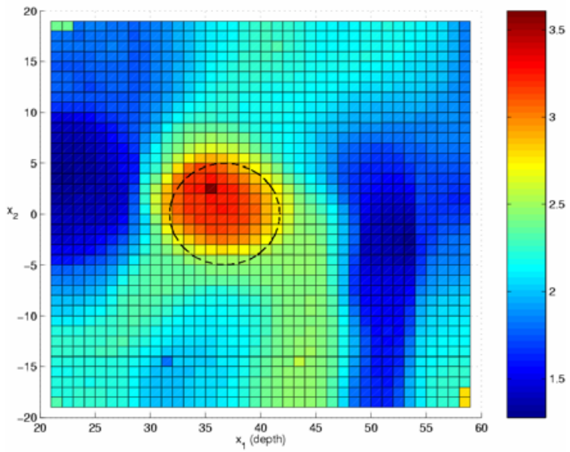
(c) $\psi_t(x,t)|_{t=\hat{T}(x)} \geq 0$;

(d) Show $\psi_t = \sqrt{\mu/\rho}$

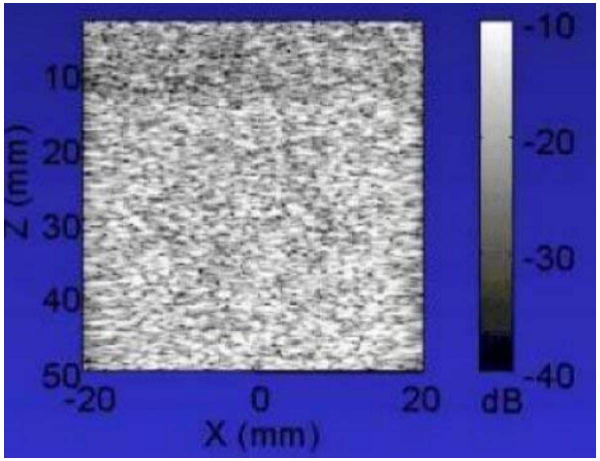




Reconstruction By Distance Method



Reconstruction By Level Curve Method



Ultrasound Image

Back Up : We have to find ϕ or $\widehat{T}$

I. Mathematically $\hat{u} = M(x)e^{i\phi(x)}$

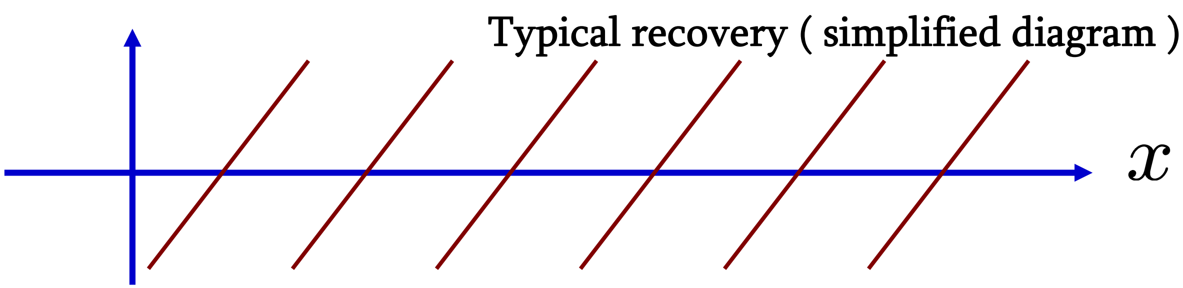
$$M = |\hat{u}|$$

$$\text{Re } \hat{u}/M = \cos\phi$$

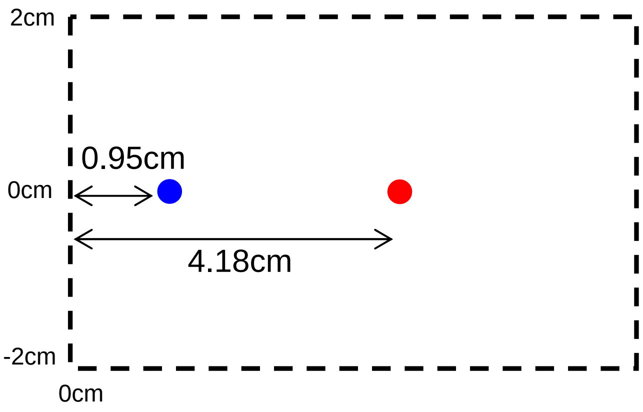
$$\text{Re } \hat{u}/ \text{ Im } \hat{u} = \cot\phi$$

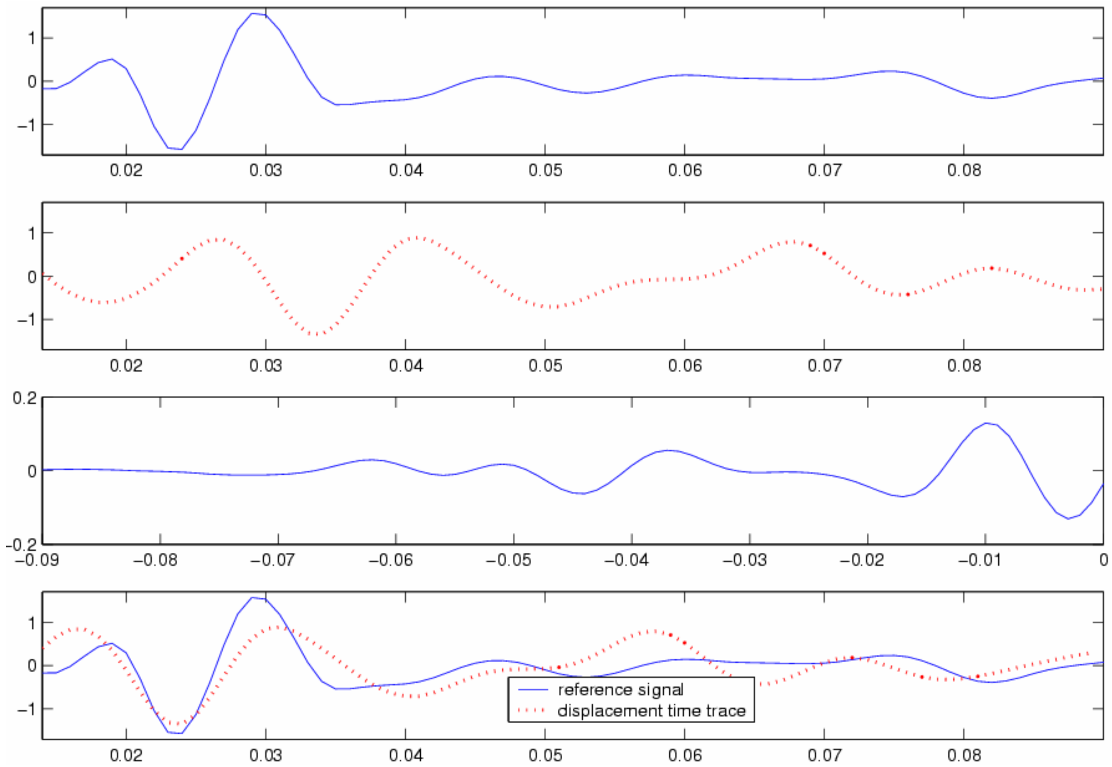
II. Phase unwrapping :

$$\phi(x) = \text{arccot}[\text{Re}\hat{u}/\text{Im}\hat{u}]$$



For $\hat{T}$ - find position of the wave packet at front of wave.



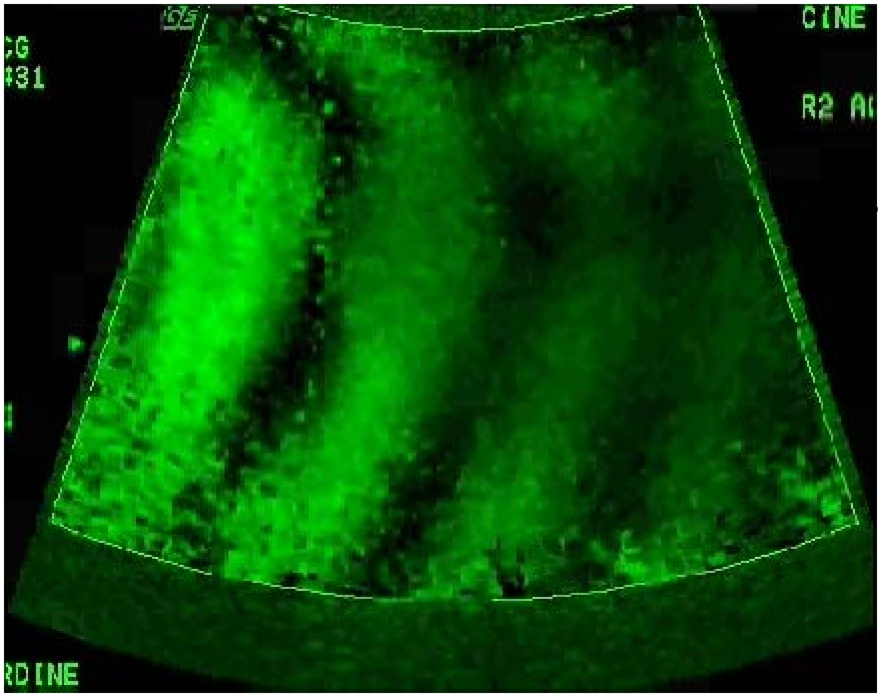


Method 1 : Determine arrival time surface by correlation

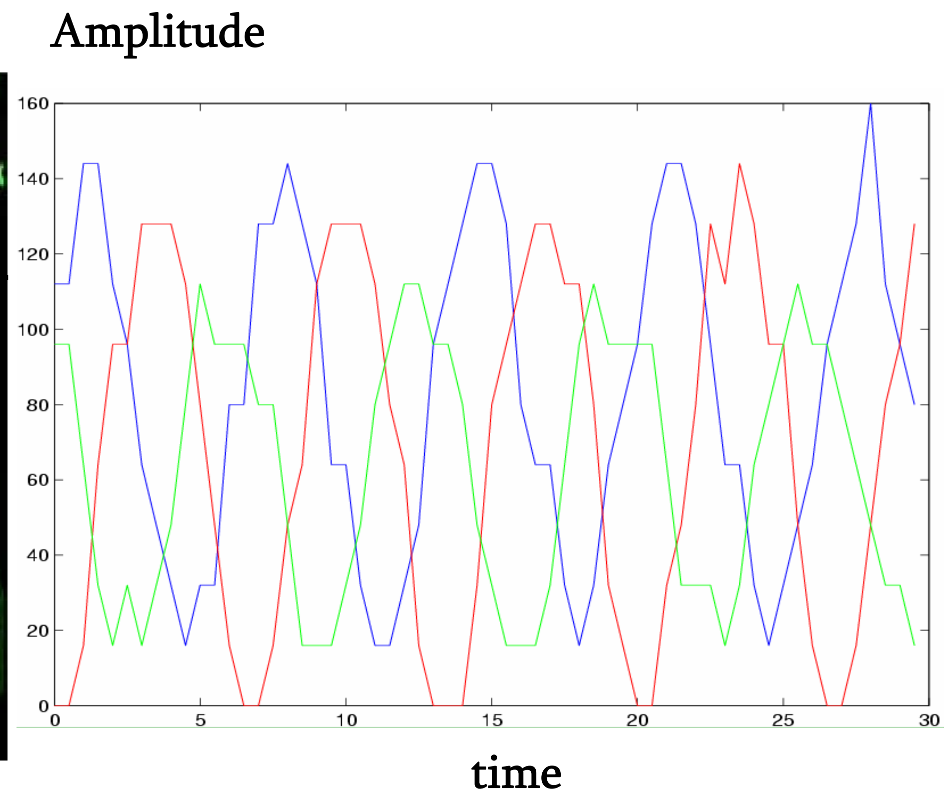
$$\hat{T}(x_1) := \hat{T}(x_0) + \Delta\hat{t} \text{ where } x_1 = x_0 + \Delta x \text{ and}$$

$$\Delta\hat{t} = \arg\max_{\Delta t} \int u(x_0, t) u(x_1, t - \Delta t)\, dt.$$

Method 2 : Take viscoelastic effects into account; See poster of Jens Klein.



Holographic Wave Movie



Amplitude Time Traces In Holographic Wave Experiment

Method
(McLaughlin, Renzi)

- Use cross covariance function to estimate arrival times of moving interference pattern

- Use Eikonal equation to find the speed of the moving interference pattern

$$c_S = \text{phase wave speed.}$$

Start with : Isotropic models

Use linear elasticity equation system or acoustic model

Geometric optics approximation

$$u_1 = a e^{i\omega_1(\phi(x)-t)}$$

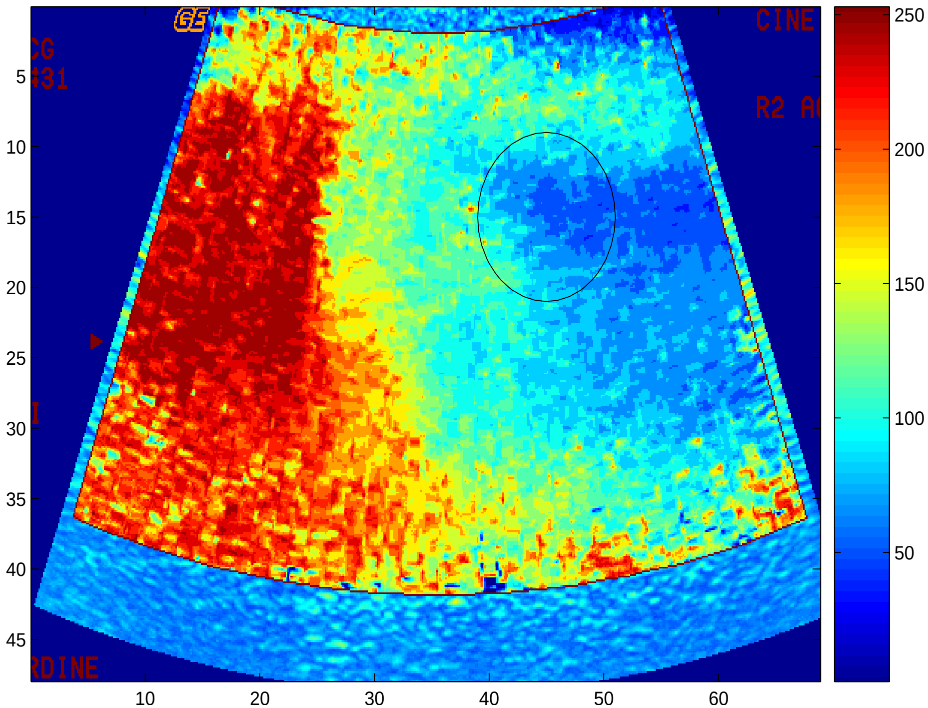
At receiver $$u_2 = b e^{-i\omega_2 t} \quad , \; \omega_1 - \omega_2 = O(10^{-1})$$

Measure $$|u| = |u_1 + u_2| = a^2 + b^2 + 2ab\cos(\psi(x,t))$$

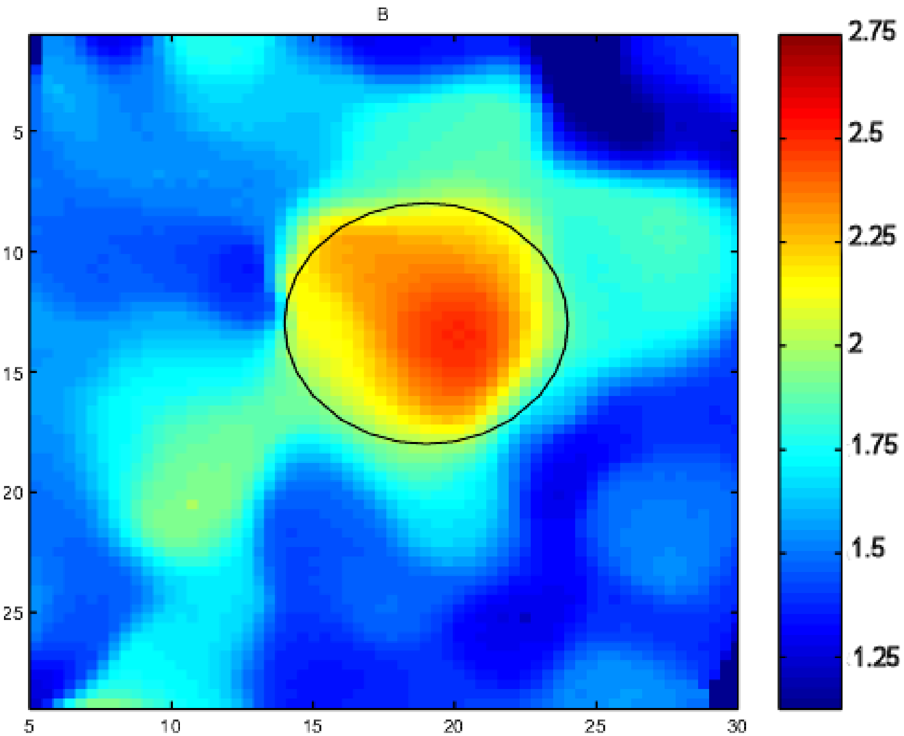
$$\gamma(x,t) = \omega_1(\phi(x) - t) + \omega_2 t, \quad \omega_2 = \omega_1 + \Delta\omega$$

$$\frac{\gamma_t^2}{|\nabla_x \gamma|^2} = \frac{(\Delta\omega)^2 c_s^2}{\omega_1^2}$$

Holographic Wave Experiment Images



Amplitude Data



Phase Wave Speed Reconstruction

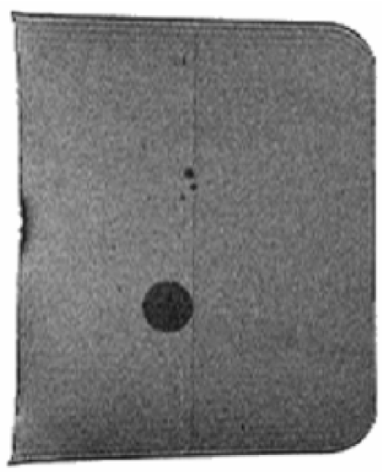
Create Hybrid Methods: Apply to MR Data Example

(McLaughlin, Renzi, Yoon)

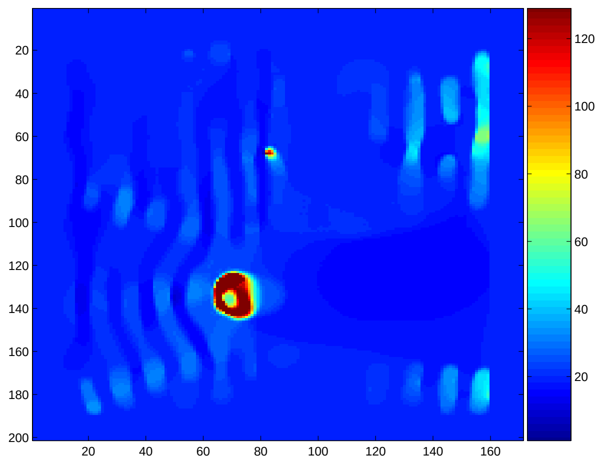
$$\mu/\rho\triangle\hat{u} + \omega^2\hat{u} = 0$$

- Example: Let $\hat{u} = Me^{i\phi}$, Assume μ/ρ is real $\Rightarrow \sqrt{\mu/\rho} = -\omega^2 M/\left[\triangle M - M|\nabla\phi|^2\right]$

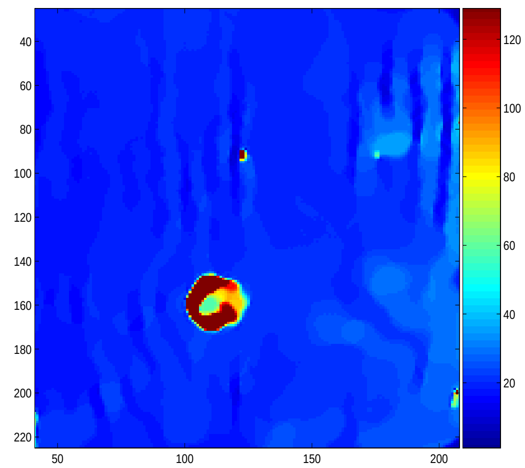
Hybrid Method
- Step 1: Calculate $\triangle M$ using finite difference averaging; Adjust step size according to SNR.
- Step 2: Calculate $|\nabla\phi|^2$ as $1/c^2$ from $c|\nabla\phi| = 1$ using level curve method ; Adjust step size according to SNR.
- Step 3: Apply local averaging to M.



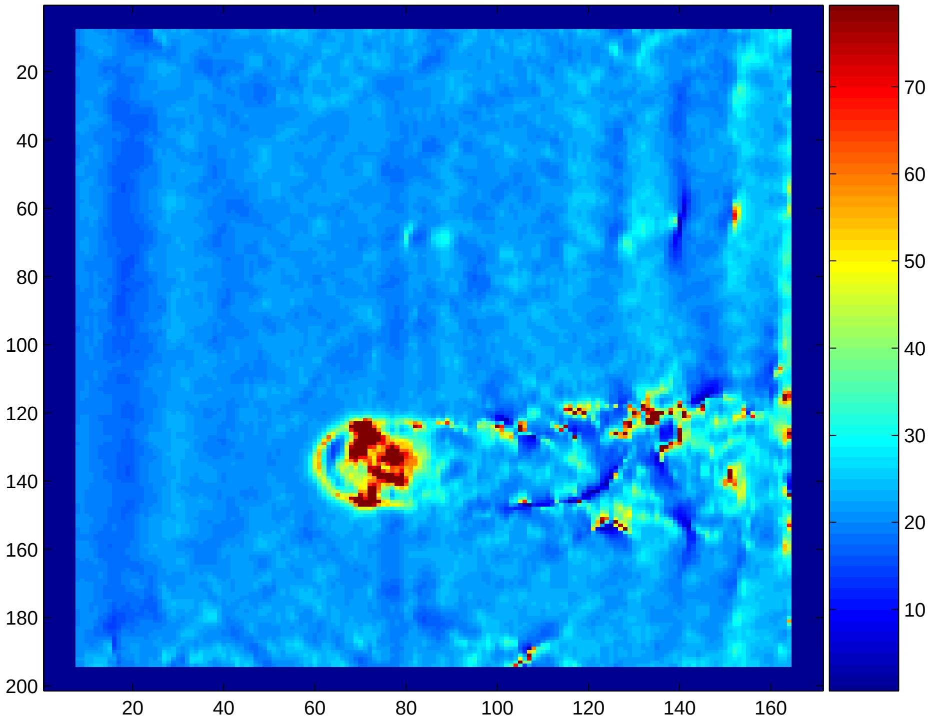
Standard MR Image



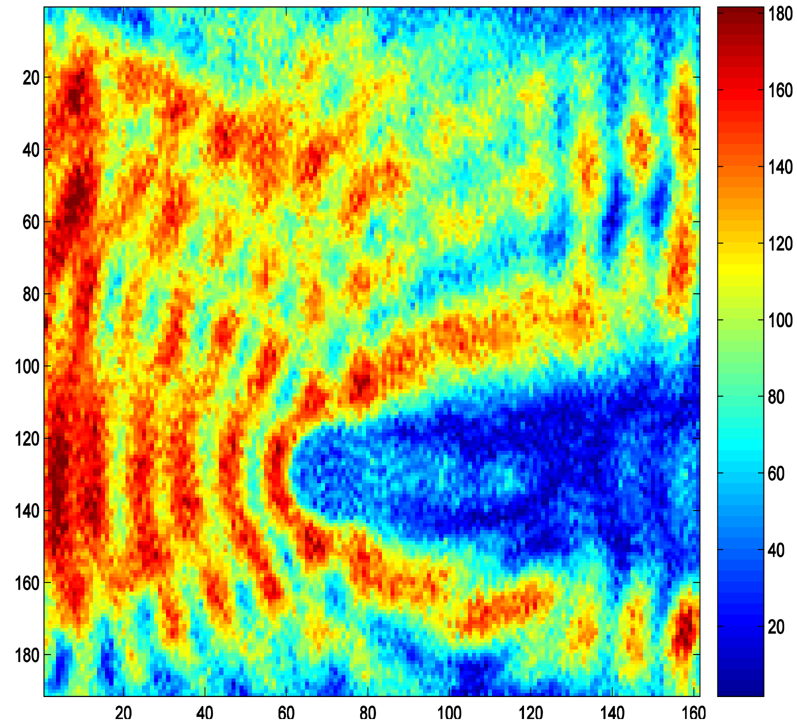
Hybrid Method



Alternate Hybrid Method



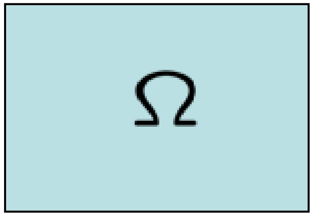
MR Data Reconstruction Without Hybrid Method



Displacement Amplitude Image

Finite Element Methods

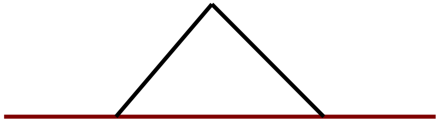
- Start with: $\nabla \cdot (\mu \nabla \hat{u}) + \omega^2 \rho \hat{u} = 0$



Weak solution formulation

$$\int_{\partial\Omega} v_j \mu \frac{d\hat{u}}{dn} - \int_{\Omega} \mu \nabla v_j \cdot \nabla \hat{u} + \int_{\Omega} \rho \omega^2 \hat{u} v_j = 0$$

$\{v_j\}_{j=1}^{n}$-test element



Advantage: Don't need to take derivatives of μ;

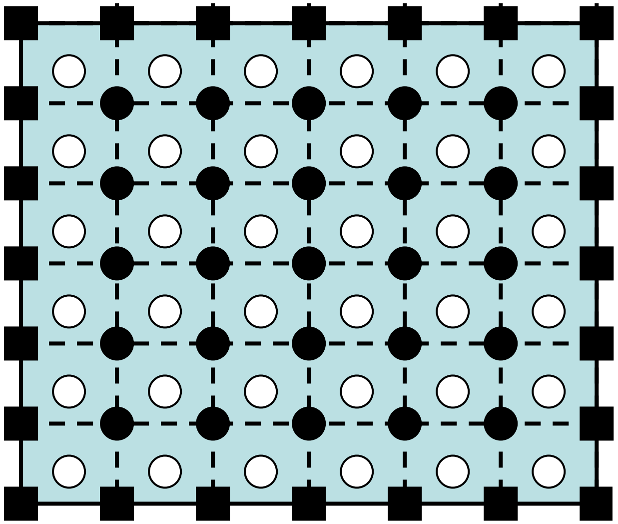
Basic Method: 1. $\widehat{u}$ and its derivatives – calculated as before;

2. Make expansions of $\mu = \sum_{i=1}^{n} a_i f_i$;

3. Find $\{a_i\}_{i=1}^{n}$ from the n equations.

Maniatty and Park (Rensselaer) – see talks at conference

- Subzone Method:



A. Eliminate boundary term;
(only for acoustic model)

B. Assume μ does not vary in normal direction in outer layer;

C. Assume $\rho \equiv$ constant;

D. Make finite element expansion for μ .

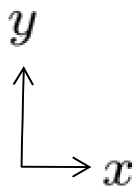
Full Elastic Model

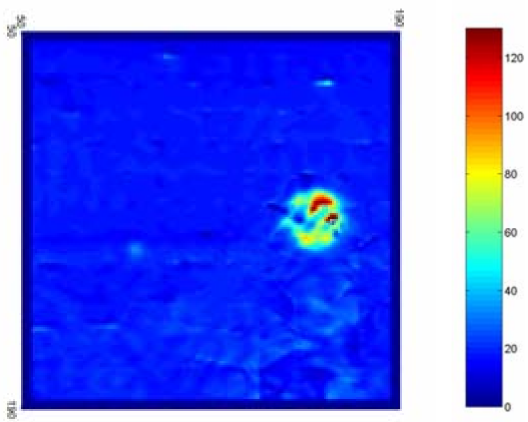
- We obtain

$$\int_\Omega p\nabla\cdot v_j+\frac{1}{2}\int_\Omega \mu\left[\nabla\hat{u}+(\nabla\hat{u})^T\right]:\left[\nabla v_j+(\nabla v_j)^T\right]+\omega^2\int_\Omega \rho\hat{u}\cdot v$$

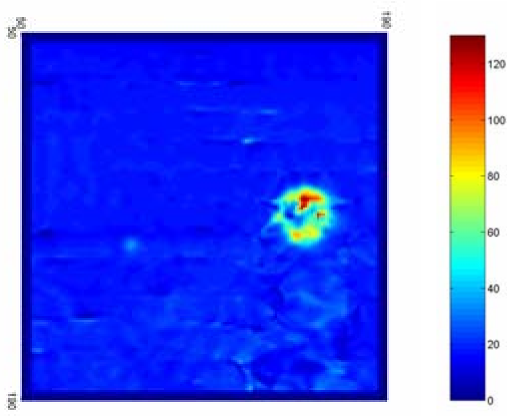
$$=\int_{\partial\Omega}\left[p\frac{dv}{dn}+\mu v_j\cdot[\nabla\hat{u}+(\nabla\hat{u})^T]\cdot n\right]d\sigma=\int_{\partial\Omega}\underset{\text{traction}}{\underset{\uparrow}{T}}v_j d\sigma$$

Apply subzone method : Now find p = pressure and μ.





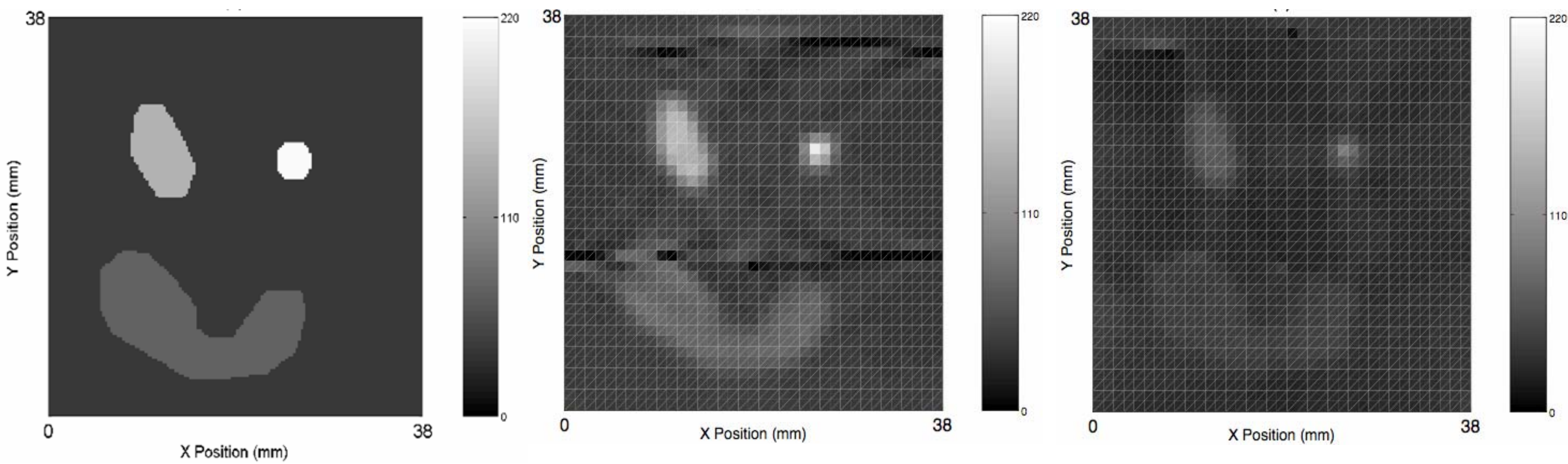
Using 3 component data



Using 2 component data (u_x,u_z), reconstruct 3rd component using incompressibility assumption with filter

Pressure Isn't Zero In Heterogeneous Case

- Recovery of μ with synthetic data – multiple inclusion;



Exact μ

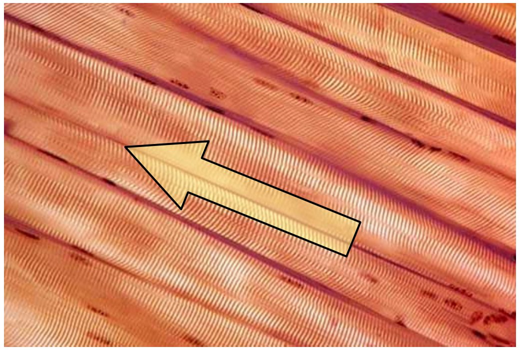
Reconstruction of p, μ with no noise;

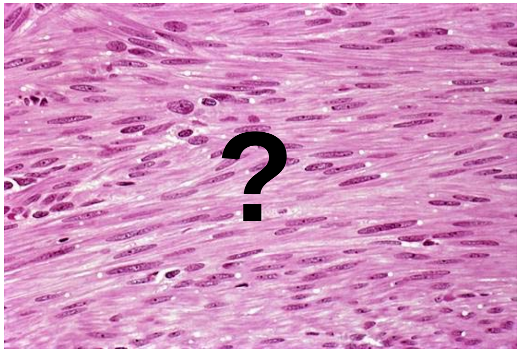
Image is of μ

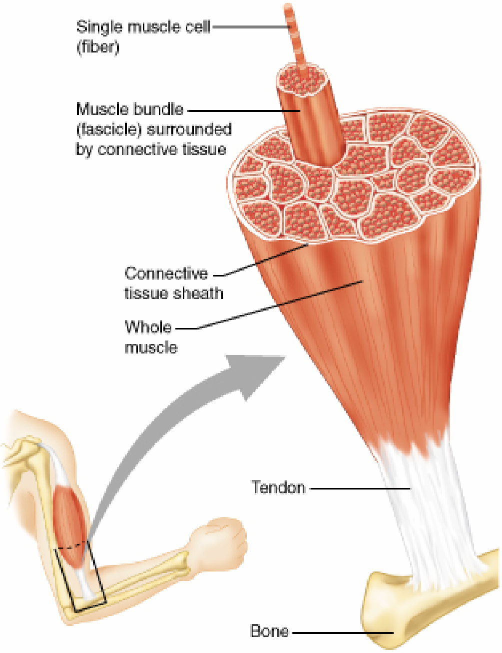
Note: $p \neq 0$

Reconstruction of μ neglecting p

Consider Anisotropic Models







- Muscle tissue has fibers ;
- Elastic properties of cancerous tissue may be anisotropic (preliminary studies) ;
- More correct models give better results ;
- Distinguish cancerous from benign tissue .

Model for Transversely Isotropic Medium

(fiber in x_3 direction)

$$\boxed{\rho \vec{u}_{tt} = \nabla \cdot \sigma = \nabla \cdot (\mathcal{C}\epsilon)} \qquad \epsilon = \frac{1}{2}(\nabla \vec{u} + \nabla \vec{u}^T), \, \mathcal{C} = (c_{ijkl})$$

By relabelling $(11, 22, 33, 23, 31, 12) \to (1, 2, 3, 4, 5, 6)$, $\sigma = \mathcal{C}\epsilon$ is represented by

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{11} - 2c_{66} & c_{13} & 0 & 0 & 0 \\ c_{11} - 2c_{66} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{66} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix}.$$

$$\boxed{\begin{aligned} &c_{11} = (1 - n\nu_{fp}^2)D, \quad c_{13} = \nu_{fp}(1 + \nu_p)D, \quad c_{33} = \frac{1}{n}(1 - \nu_p^2)D, \\ &c_{44} = \mu_{fp}, \quad c_{66} = \frac{E_p}{2(1 + \nu_p)}, \quad D := \frac{E_p}{(1 + \nu_p)(1 - \nu_p - 2n\nu_{fp}^2)}. \end{aligned}}$$

E_p, E_f =Young's moduli
ν_p, ν_{fp} =Poisson's ratios
μ_{fp} =shear modulus
$n = E_p/E_f$

Transversely Isotropic Medium

Incompressible condition: $0 = \nabla \cdot \vec{u} = \mathrm{tr}(\epsilon)$ $\epsilon = \frac{1}{2}(\nabla\vec{u} + \nabla\vec{u}^T)$

For transversely isotropic medium: $1 = \nu_p + n\nu_{fp}, \quad 1 = 2\nu_{fp}$

Identify

E_p, E_f =Young's moduli

μ_{fp} =shear modulus

$\vec{f}$ = fiber direction

Or identify

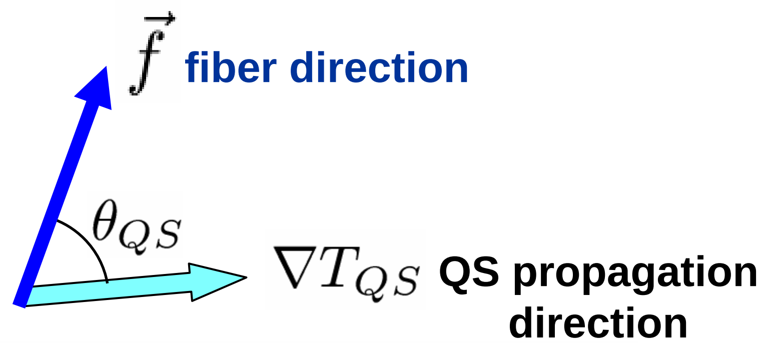
$c_{44} = \mu_{fp}$

$$c_{66} = \frac{E_f E_p}{(4E_f - E_p)}$$

$n = E_p/E_f$

$\vec{f}$ = fiber direction

Incompressible Arrival Time Or Phase Equations



$$c_{SH}^2 = \frac{c_{66}}{\rho} + \left(\frac{c_{44}}{\rho} - \frac{c_{66}}{\rho}\right) \frac{(\nabla T_{SH} \cdot \vec{f})^2}{|\nabla T_{SH}|^2}$$

$$c_{QS}^2 = \frac{c_{66}}{\rho n} + \left(\frac{c_{44}}{\rho} - \frac{c_{66}}{\rho n}\right) \left(1 - 2(\nabla T_{QS} \cdot \vec{f})^2 / |\nabla T_{QS}|^2\right)^2$$

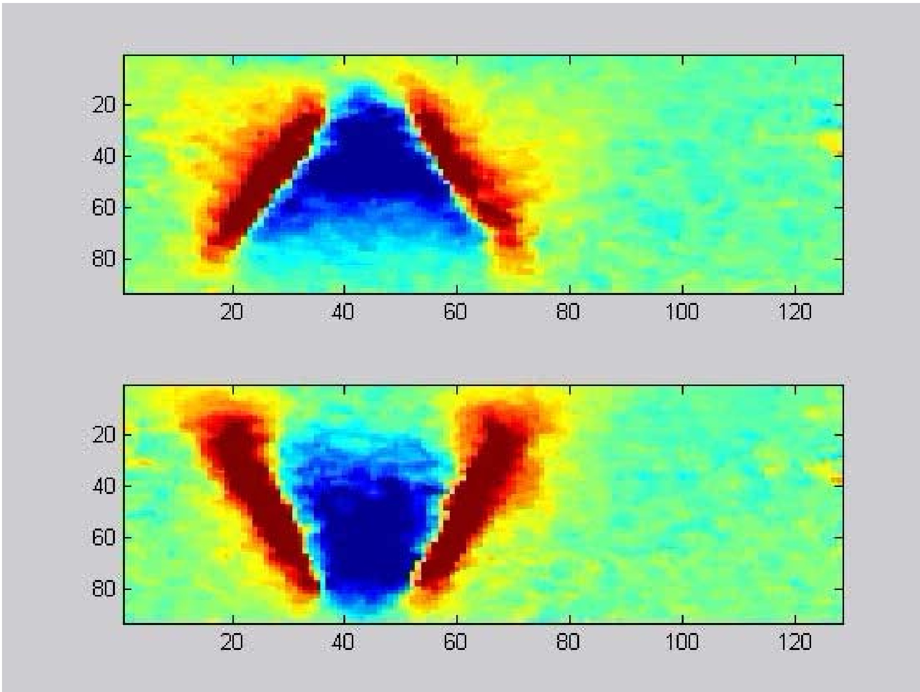
Goal: Find $E_p, E_f, \mu_{fp},$ and $\vec{f}$

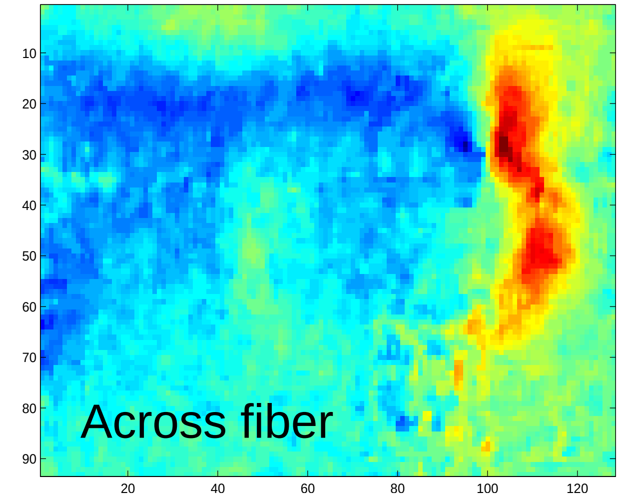
Recall $c_{44} = \mu_{fp}$, $c_{66} = \dfrac{E_f E_p}{(4E_f - E_p)}$, $n = E_p / E_f$

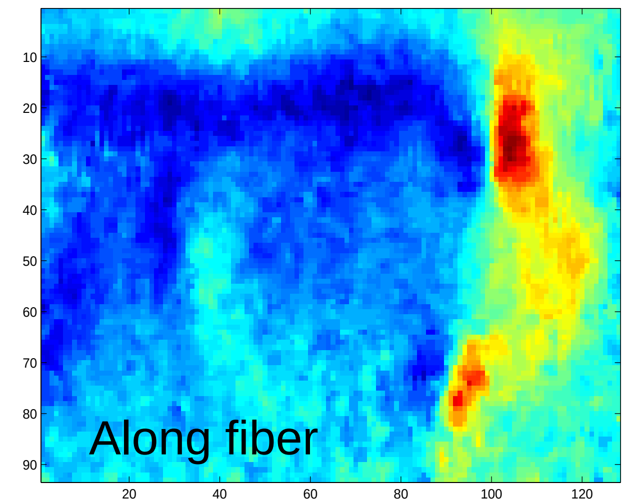
c_{SH}, c_{QS} - speed of SH (or QS) wave in direction orthogonal to lines of constant phase.

Wave Fronts in Measured Data

Fiber direction



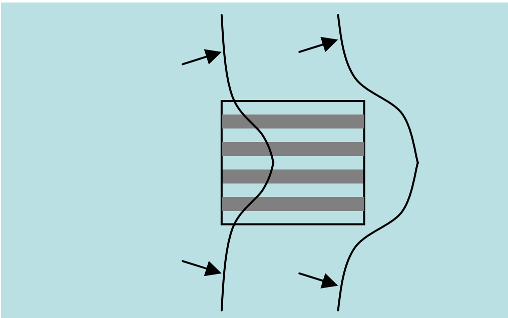




$$c_{SH}^2 = \frac{c_{66}}{\rho} + \left(\frac{c_{44}}{\rho} - \frac{c_{66}}{\rho}\right)\left(\nabla\psi_{SH} \cdot \vec{f}\right)$$

$$\nabla\psi_{SH} = \nabla\hat{T} / |\nabla\hat{T}_{SH}|$$

Anisotropic Case: Propagating Fronts

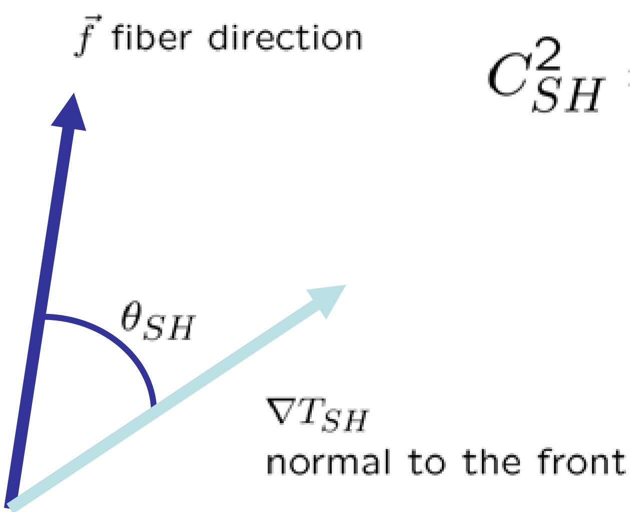


Arrival Time:

$$T_{SH}(x) = \inf\{t \in (0,T) : u(x,s) = 0, s \leq t\}$$

Theorem: If C_{44}, C_{66}, $\rho \in C^1(\bar{\Omega})$ and T_{SH} is Lipschitz continuous, then

$$C_{SH}^2 = \frac{C_{66}}{\rho} + \left(\frac{C_{44}}{\rho} - \frac{C_{66}}{\rho}\right) \frac{(\nabla T_{SH} \cdot \vec{f})^2}{|\nabla T_{SH}|^2}.$$



$$C_{SH}^2 |\nabla T_{SH}|^2 = 1.$$

Goal 1: Given T_{SH} find C_{SH}.

In Isotropic and Anisotropic Cases:

Goal 1: Find speed in direction orthogonal to front;

Anisotropic Case:

Goal A: Given $\vec{f}$, C_{66}/ρ and $\nabla T_{SH} \not\perp \vec{f}$, image $\sqrt{C_{44}/\rho}$;

Goal B: Given $\vec{f}$, $\nabla T_{SH} \perp \vec{f}$, image $\sqrt{C_{66}/\rho}$;

Goal C: Given multiple T_{SH}, find $\sqrt{C_{44}/\rho}$, $\sqrt{C_{66}/\rho}$, $\vec{f}$.

Isotropic: $C_S|\nabla \hat{T}| = \sqrt{\frac{\mu}{\rho}}|\nabla \hat{T}| = 1$

Anisotropic: $1 = C_{SH}^2|\nabla T_{SH}|^2 = \frac{C_{66}}{\rho}|\nabla T_{SH}|^2 + \left(\frac{C_{44}}{\rho} - \frac{C_{66}}{\rho}\right)(\nabla T_{SH} \cdot \vec{f})^2$

We can find wave speed in each case

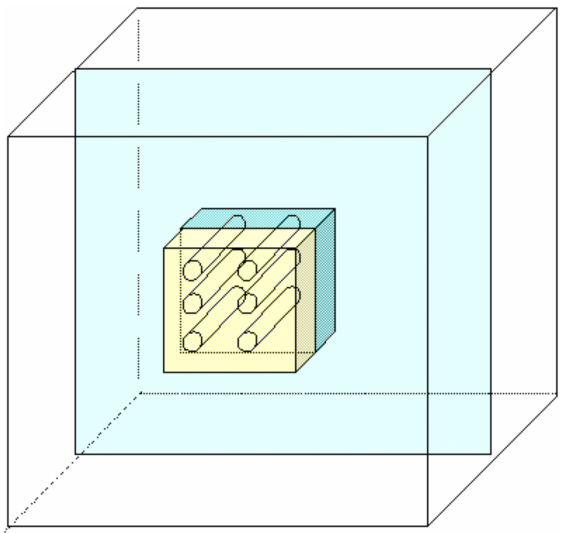
Image plane perpendicular to fibers

(Single component measurement, downward direction)

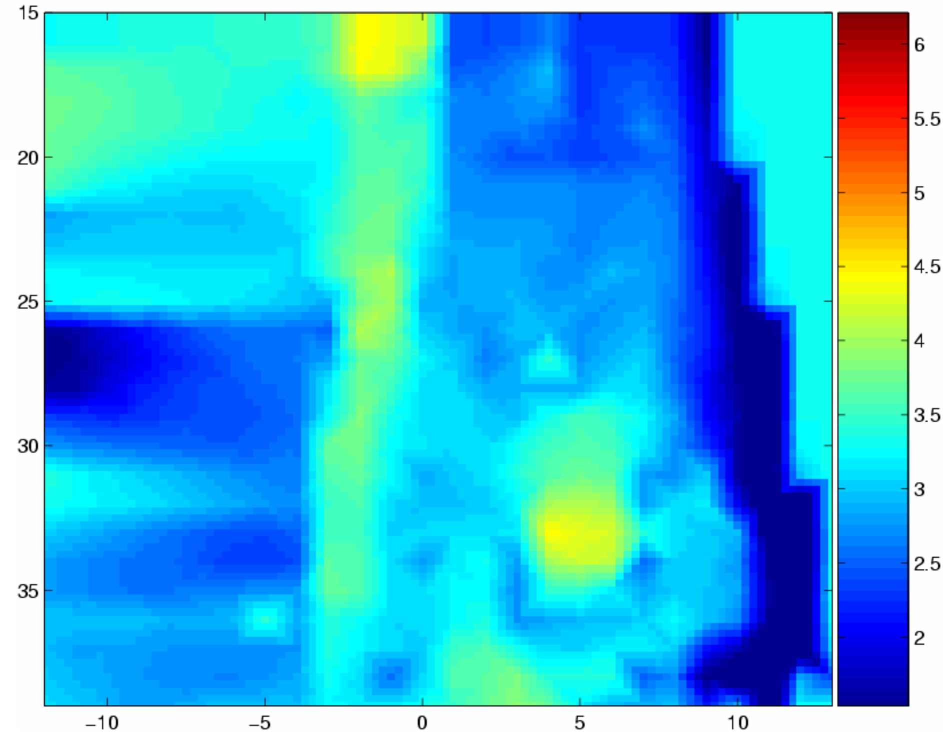
0

$$c_{SH}^2 |\nabla T_{SH}|^2 = \dot{\mathrm{i}} = \frac{c_{66}}{\rho} |\nabla T_{SH}|^2 + \left(\frac{c_{44}}{\rho} - \frac{c_{66}}{\rho} \right) (\nabla T_{SH} \cdot \vec{f})^2$$

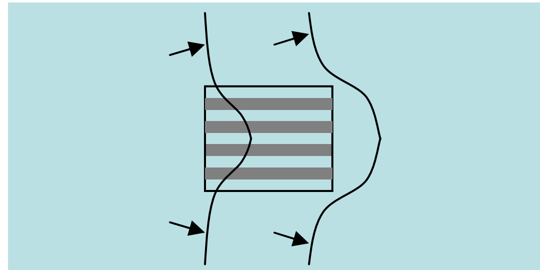
$$\Rightarrow \quad c_{SH}^2 = \frac{c_{66}}{\rho}$$

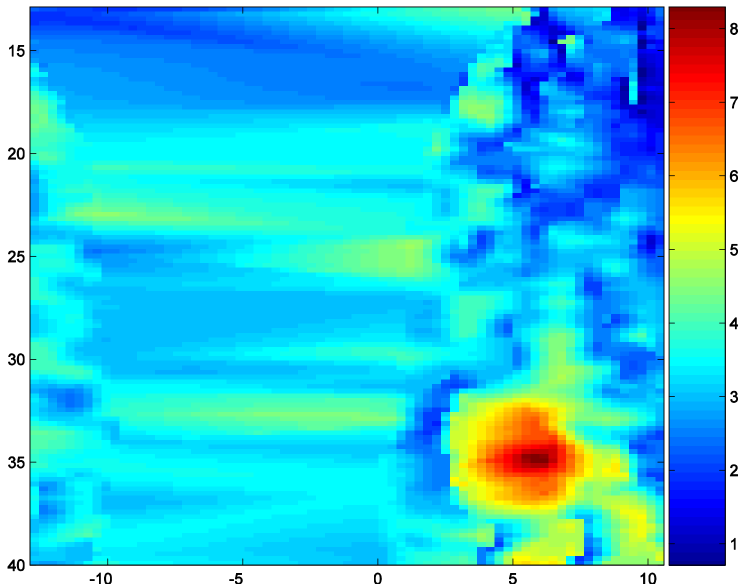


Single measurement



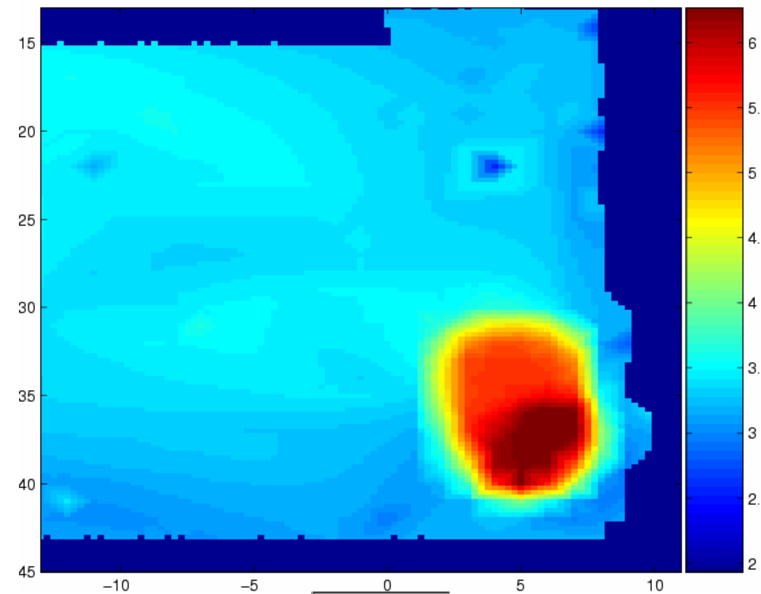
Single Measurement – Downward Component





Wave Speed Normal

To The Wave Front



$\sqrt{C_{44}/\rho}$

Assume $\vec{f}$ and C_{66}/ρ known

3D - Basic Equation

$$C_{SH}^2 = \phi_t^2 = \frac{C_{66}}{\rho} + \left(\frac{C_{44}}{\rho} - \frac{C_{66}}{\rho}\right)(\nabla_x \psi \cdot \vec{f})^2 \quad t = T_{SH}(x)$$

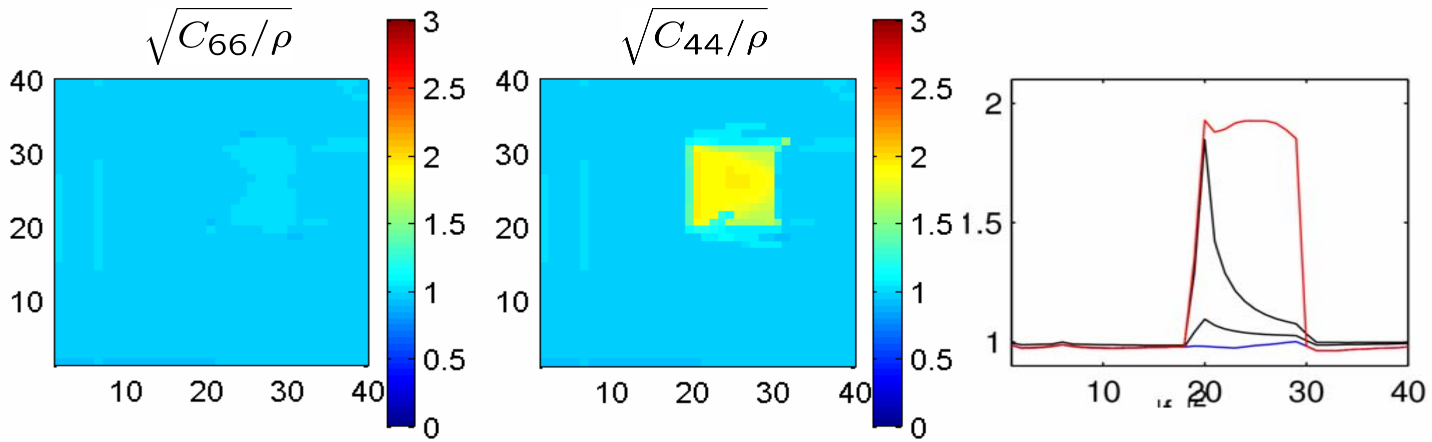
With four data sets $\{(\nabla_x \psi^j, C_{SH}^j)\}_{j=1}^4$, there are at most four distinct solutions $\{C_{44}^k/\rho, C_{66}^k/\rho, \vec{f}^k\}_{k=1}^4$.

(This is a consequence of the nonlinearity)

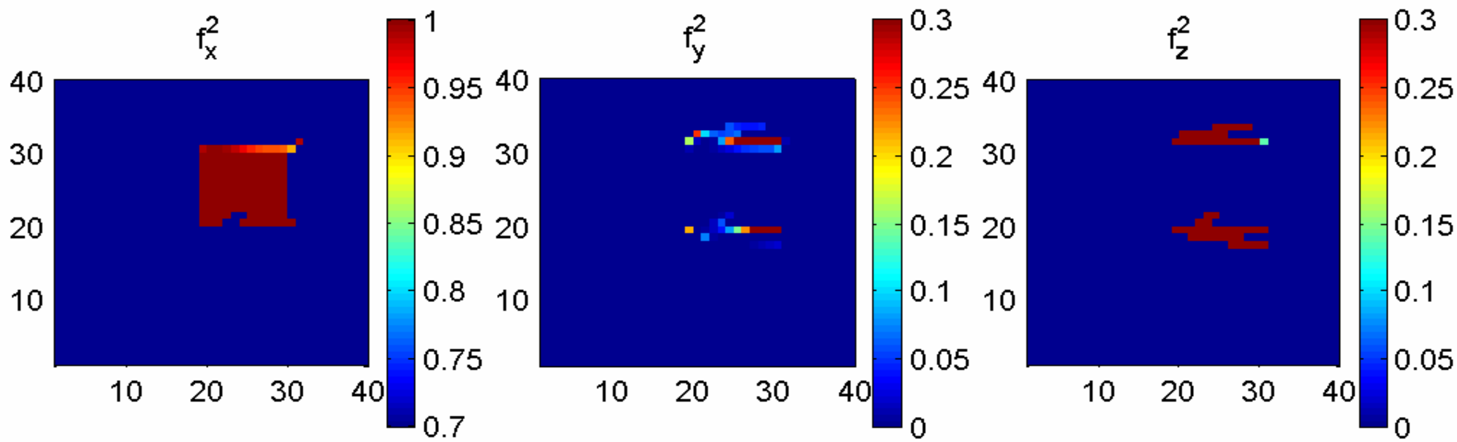
Basic Property:

C_{66}/ρ is the root of a fourth degree polynomial.

Numerical Reconstructions With Synthetic Data



[True $c_{44}/\rho \equiv 1$] [True $\sqrt{c_{44}/\rho} = 2$ in inclusion]



[True $f_x^2 = 1$ in inclusion][True $f_y^2 = 0$ in inclusion][True $f_z^2 = 0$ in inclusion]

Total Variation Denoising

- For all images – can apply smoothing to images.

$$\min_{\sqrt{\mu/\rho}\in \mathrm{BV}} \left[\gamma \int_\Omega \left\{ \underbrace{\left(\sqrt{\frac{\mu}{\rho}}\right)_D}_{\text{Values from reconstruction}} - \left(\sqrt{\frac{\mu}{\rho}}\right)_F \right\}^2 + \int_\Omega \left| \nabla \left(\sqrt{\frac{\mu}{\rho}}\right)_F \right| \right]$$

Values from reconstruction **Targeted final imaging functional**

with regularized Euler-Lagrange equation

$$\nabla \cdot \left[\frac{\nabla \left(\sqrt{\frac{\mu}{\rho}}\right)_F}{\left|\nabla \left(\sqrt{\frac{\mu}{\rho}}\right)_F\right| + \epsilon} \right] + 2\gamma \left[\left(\sqrt{\frac{\mu}{\rho}}\right)_D - \left(\sqrt{\frac{\mu}{\rho}}\right)_F \right] = 0$$

Introducing the evolving variable s, solve

$$\frac{\partial}{\partial s}\sqrt{\frac{\mu}{\rho}} = \nabla \cdot \left[\frac{\nabla \left(\sqrt{\frac{\mu}{\rho}}\right)_F}{\left|\nabla \left(\sqrt{\frac{\mu}{\rho}}\right)_F\right| + \epsilon} \right] + 2\gamma \left[\left(\sqrt{\frac{\mu}{\rho}}\right)_D - \left(\sqrt{\frac{\mu}{\rho}}\right)_F \right] \qquad \left.\sqrt{\frac{\mu}{\rho}}\right|_{s=0} = \left(\sqrt{\frac{\mu}{\rho}}\right)_D$$

Conclusion

- Inverse Problem:
  1. Modeling;
  2. Identify richest or most usable data sets;
  3. Identify acceptable approximations;
  4. Smart algorithms.
- Data:
  1. Data smoothing that maximizes information content;
  2. Identify rich data subsets.
- Algorithm Options Covered In This Talk:
  1. Finite Difference based local algorithms;
  2. Arrival Time algorithms;
  3. Finite Element weak solution formulations.